Engineering Notes

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Solution of Linear Gyroscopic Systems

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I. Introduction

THE solution of gyroscopic system equations requires solution of an eigenvalue problem which by present methods is accomplished via a transformation to a standard non-gyroscopic eigenvalue problem on two symmetric matrices with its associated computational advantages. However this is done at the expense of doubling the dimension and simultaneously doubling the multiplicities of the eigenfrequencies. In this paper a method is presented which, for most practical problems eliminates these difficulties. The method takes advantage of special phase relationships satisfied in many practical systems.

II. Reducible Gyroscopic Systems

Consider a rotating flexible structure that has a stable dynamical equilibrium consisting of a uniform rotation. After discretization of the flexible elements, the Lagrangian formalism for the free vibrations about the dynamical equilibrium takes the following form:

$$M\ddot{q}+G\dot{q}+Kq=0 \tag{1}$$

where $M=M^T$, $G=-G^T$, and $K=K^T$ are real, constant, $n\times n$ matrices; and $q(t)$ $(n\times 1)$ describes the displacements of a body-fixed frame and the elastic displacements.

For a stable system which does not require $K>0$ the solution $q(t)$ of Eq. (1) consists only of harmonic oscillations, and $q(t)$ becomes

$$q(t)=x\sin\omega t+y\cos\omega t \tag{2}$$

where x, y are $(n\times 1)$ constant matrices. They contain the amplitude and relative phase information of the vibrations at frequency ω.

First, we consider the special case where Eq. (1) can be partitioned as

$$\begin{bmatrix} M_p & 0 \\ 0 & M_q \end{bmatrix}\ddot{q}+\begin{bmatrix} 0 & G \\ -G^T & 0 \end{bmatrix}\dot{q}+\begin{bmatrix} K_p & 0 \\ 0 & K_q \end{bmatrix}q=0 \tag{3}$$

where the index indicates the dimensions of the square submatrices, and G is rectangular $(p\times q)$. Many gyroscopic systems that arise in practice have this particular structure.

Partitioning $q^T[q_p^T q_q^T]$, $x^T[x_p^T x_q^T]$, and $y^T[y_p^T, y_q^T]$, we substitute a trial solution of the form of Eq. (2) into Eq. (3). Grouping terms in $\sin\omega t$ and $\cos\omega t$ one obtains

$$[-\omega^2 M_p x_p-\omega G y_q+K_p x_p]\sin\omega t + [-\omega^2 M_p y_p+\omega G x_q+K_p y_p]\cos\omega t=0 \tag{4}$$

$$[-\omega^2 M_q x_q+\omega G^T y_p+K_q x_q]\sin\omega t + [-\omega^2 M_q y_q-\omega G^T x_p+K_q y_q]\cos\omega t=0 \tag{5}$$

The p equations (4) and the q equations (5) have to be satisfied for all t. This is only possible if the coefficients of $\sin\omega t$ and $\cos\omega t$ vanish. The four corresponding conditions grouped in two $n\times n$ matrix equations are

$$\begin{bmatrix} -\omega^2 M_p+K_p & -\omega G \\ -\omega G^T & -\omega^2 M_q+K_q \end{bmatrix}\begin{bmatrix} x_p \\ y_q \end{bmatrix}=0 \tag{6}$$

$$\begin{bmatrix} -\omega^2 M_p+K_p & \omega G \\ \omega G^T & -\omega^2 M_q+K_q \end{bmatrix}\begin{bmatrix} y_p \\ x_q \end{bmatrix}=0 \tag{7}$$

and Eq. (7) is easily rewritten as

$$\begin{bmatrix} -\omega^2 M_p+K_p & -\omega G \\ -\omega G^T & -\omega^2 M_q+K_q \end{bmatrix}\begin{bmatrix} y_p \\ -x_q \end{bmatrix}=0 \tag{8}$$

Now, the two homogeneous systems, Eqs. (6) and (8), are the same. The frequencies ω_i are the zeros of

$$\det\begin{bmatrix} -\omega^2 M_p+K_p & -\omega G \\ -\omega G^T & -\omega^2 M_q+K_q \end{bmatrix}=0$$

Let $S_i^T[S_{pi}^T S_{qi}^T]$ be the nontrivial solution corresponding to ω_i, then the solution $q_i(t)$ becomes

$$q_{pi}=S_{pi}(\sin\omega_i t+\cos\omega_i t)=(\sqrt{2}/2)S_{pi}\sin(\omega_i t+45)$$
$$q_{qi}=S_{qi}(-\sin\omega_i t+\cos\omega_i t)=(\sqrt{2}/2)S_{qi}\cos(\omega_i t+45)$$

As the origin of the phasing is arbitrary and the solution S_i is only defined up to a multiplicative constant, we have the complete solution of the partitioned eigenvalue problem by solving the following symmetric, real, $n\times n$ eigenvalue problem:

$$\begin{bmatrix} -\omega^2 M_p+K_p & -\omega G \\ -\omega G^T & -\omega^2 M_q+K_q \end{bmatrix}\begin{bmatrix} S_p \\ S_q \end{bmatrix}=0 \tag{9}$$

The first part (S_p) of the eigenvector contains the sine terms and the second part (S_q) the cosine terms. The physical meaning of the partitioned structure, Eq. (3), is to give the phasing information between the two sets of components q_p and q_q. The eigenvector S_i gives the relative amplitudes of the vibrations at frequency ω_i.

The eigenvalue problem, Eq. (9), is still a nonstandard eigenvalue problem due to the presence of the linear term in ω. These terms can be eliminated by including ω in S_q and premultiplying Eq. (8) by

$$\begin{bmatrix} E_p & 0 \\ 0 & \omega E_q \end{bmatrix}$$

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One obtains

$$-\omega^2\begin{bmatrix} M_p & 0 \\ G^T & M_q \end{bmatrix}\begin{bmatrix} S_p \\ \omega S_q \end{bmatrix}+\begin{bmatrix} K_p & -G \\ 0 & K_q \end{bmatrix}\begin{bmatrix} S_p \\ \omega S_q \end{bmatrix}=0 \tag{10}$$

Equation (10) represents a standard generalized eigenvalue problem $(A-\lambda B)x'=0$ on two nonsymmetric matrices. To transform Eq. (10) into an eigenvalue problem on two symmetric matrices, we first eliminate G^T.

Premultiplying Eq. (10) by $\begin{bmatrix} 0 & 0 \\ G^T M_p^{-1} & 0 \end{bmatrix}$, we have

$$-\omega^2\begin{bmatrix} 0 & 0 \\ G^T & 0 \end{bmatrix}\begin{bmatrix} S_p \\ \omega S_q \end{bmatrix} \tag{11}$$

$$+\begin{bmatrix} 0 & 0 \\ G^T M_p^{-1} K_p & -G^T M_p^{-1} G \end{bmatrix}\begin{bmatrix} S_p \\ \omega S_q \end{bmatrix}=0$$

Equation (11) can be used in Eq. (10) to eliminate G^T:

$$-\omega^2\begin{bmatrix} M_p & 0 \\ 0 & M_q \end{bmatrix}\begin{bmatrix} S_p \\ \omega S_q \end{bmatrix} \tag{12}$$

$$+\begin{bmatrix} K_p & -G \\ -G^T M_p^{-1} K_p & K_q+G^T M_p^{-1} G \end{bmatrix}\begin{bmatrix} S_p \\ \omega S_q \end{bmatrix}=0$$

Then changing to the variables $t_p=M_p^{-1}K_pS_p$, $t_q=\omega S_q$, gives Eq. (12) a symmetric form:

$$-\omega^2\begin{bmatrix} M_p K_p^{-1} M_p & 0 \\ 0 & M_q \end{bmatrix}\begin{bmatrix} t_p \\ t_q \end{bmatrix}$$

$$+\begin{bmatrix} M_p & -G \\ -G^T & K_q+G^T M_p^{-1} G \end{bmatrix}\begin{bmatrix} t_p \\ t_q \end{bmatrix}=0 \tag{13}$$

With Eq. (13), the gyroscopic problem (3) is completely reduced to a nongyroscopic problem and one of the same dimension rather than the usual doubling of the dimension. Gyroscopic systems with the structure of Eq. (3) are in a reducible form. By stretching the language, they may be called reducible systems.

Most of the gyroscopic problems encountered in practice have the structure of Eq. (3) and hence are reducible. Some examples are: an asymmetric rigid body spinning uniformly around its stable axis, appended or not with a flexible axial boom, such a body appended with two radial wires attached in meridian plane containing the spin axis and a second principal axis.[4] Other examples given in Ref. 2 are gravity-gradient stabilization of a spinning satellite, two spring-connected gravity-gradient stabilized rigid bodies, and elastically suspended rotors.

Notice that Eq. (12) can be put in a symmetric form in a different way.

Premultiplying by $\begin{bmatrix} K_p M_p^{-1} & 0 \\ 0 & E_q \end{bmatrix}$,

$$-\omega^2\begin{bmatrix} K_p & 0 \\ 0 & M_q \end{bmatrix}\begin{bmatrix} S_p \\ \omega S_q \end{bmatrix} \tag{14}$$

$$+\begin{bmatrix} K_p M_p^{-1} K_p & -K_p M_p^{-1} G \\ -G^T M_p^{-1} K_p & K_q+G^T M_p^{-1} G \end{bmatrix}\begin{bmatrix} S_p \\ \omega S_q \end{bmatrix}=0$$

Now the equivalent mass matrix is simpler and the new stiffness matrix more complicated.

Equations (13) and (14) both require $K_p>0$ for further diagonalization. If this condition is not met, an expression like Eq. (14), containing only K_q in the mass-matrix, can be obtained by symmetry.

$$-\omega^2\begin{bmatrix} M_p & 0 \\ 0 & K_p \end{bmatrix}\begin{bmatrix} \omega S_p \\ S_q \end{bmatrix} \tag{15}$$

$$+\begin{bmatrix} K_p-GM_q^{-1}G^T & GM_q^{-1}K_q \\ K_qM_q^{-1}G^T & K_qM_q^{-1}K_q \end{bmatrix}\begin{bmatrix} \omega S_p \\ S_q \end{bmatrix}=0$$

Although $K>0$ is not required, one should have $K_p>0$ or $K_q>0$.

Equations (14) and (15) can also be obtained with an elegant trick used by Meirovitch.[5] Add to the system of Eq. (1) the identity

$$K\dot{q}-K\dot{q}=0 \tag{16}$$

to immediately obtain an eigenvalue problem on a symmetric and antisymmetric matrix $(2n\times 2n)$:

$$\begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \dot{q} \end{bmatrix}+\begin{bmatrix} G & K \\ -K & 0 \end{bmatrix}\begin{bmatrix} \dot{q} \\ q \end{bmatrix}=0 \tag{17}$$

Equation (17) is then converted into the following nongyroscopic eigenvalue problem.[5]

$$-\omega^2\begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} \omega x \\ y \end{bmatrix} \tag{18}$$

$$+\begin{bmatrix} G^TM^{-1}G+K & G^TM^{-1}K \\ KM^{-1}G & KM^{-1}K \end{bmatrix}\begin{bmatrix} \omega x \\ y \end{bmatrix}=0$$

Note that $K>0$ is a requirement for the simultaneous diagonalization of the two matrices in Eq. (18). Imposing now the structure of Eq. (3), Eq. (18) becomes

$$-\omega^2\begin{bmatrix} M_p & & & \\ & M_q & & \\ & & K_p & \\ & & & K_q \end{bmatrix}\begin{bmatrix} \omega x_p \\ \omega x_q \\ y_p \\ y_q \end{bmatrix}+\begin{bmatrix} K_p-GM_q^{-1}G^T & 0 & 0 & GM_q^{-1}K_q \\ 0 & K_q+G^TM_p^{-1}G & -G^TM_p^{-1}K_p & 0 \\ 0 & -K_pM_p^{-1}G & K_pM_p^{-1}K_p & 0 \\ K_qM_q^{-1}G^T & 0 & 0 & K_qM_q^{-1}K_q \end{bmatrix}\begin{bmatrix} \omega x_p \\ \omega x_q \\ y_p \\ y_q \end{bmatrix}=0 \tag{19}$$

Table 1

n	$n(n-1)/2$	p	$p \cdot q$
2	2	1	1
3	3	1	2
4	6	1	3
		2	4
5	10	1	4
		2	6

and Eq. (19) decouples into Eqs. (14) and (15). As explained before, the two decoupled problems defined by Eq. (19) are identical when the influence of a phase-shift is considered.

III. Reducibility of a General Gyroscopic System

The principal advantages of solving a gyroscopic system by means of Eqs. (13), (14), or (15) is to avoid the doubling of the multiplicity of the frequencies which occurs in Eq. (18). One also works systematically on matrices of smaller dimensions. As mentioned previously, most of the gyroscopic systems encountered in practice have the required partitioned structure of Eq. (3). The question still remains, however, if it is possible to transform a general gyroscopic system, Eq. (1), into a reducible gyroscopic system.

Starting from scratch, we try solutions as given by Eq. (2) in Eq. (1). The requirement that the coefficients of $\sin\omega t$ and $\cos\omega t$ should vanish for all t leads to:

$$-\omega^2 Mx - \omega Gy + Kx = 0 \tag{20}$$

$$-\omega^2 My + \omega Gx + Ky = 0 \tag{21}$$

Combining Eqs. (20) and (21) in one matrix equation and replacing $y_I = \omega y$,

$$-\omega^2 \begin{bmatrix} M & 0 \\ -G & M \end{bmatrix} \begin{bmatrix} x \\ y_I \end{bmatrix} + \begin{bmatrix} K & -G \\ 0 & K \end{bmatrix} \begin{bmatrix} x \\ y_I \end{bmatrix} = 0 \tag{22}$$

Equation (22) is a generalized eigenvalue problem on two general matrices. By using the inverse of the first of the two matrices, or by eliminating $-G$ as between Eqs. (10) and (11), it is then easily transformed into a problem on two symmetric matrices:

$$-\omega^2 \begin{bmatrix} MK^{-1}M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} x_I \\ y_I \end{bmatrix} + \begin{bmatrix} M & G^T \\ G & K+G^TM^{-1}G \end{bmatrix} \begin{bmatrix} x_I \\ y_I \end{bmatrix} = 0 \tag{23}$$

with $x = K^{-1}Mx_I$ or

$$-\omega^2 \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} x \\ y_I \end{bmatrix} + \begin{bmatrix} KM^{-1}K & KM^{-1}G^T \\ GM^{-1}K & K+G^TM^{-1}G \end{bmatrix} \begin{bmatrix} x \\ y_I \end{bmatrix} = 0 \tag{24}$$

Notice also that Eq. (22) becomes a problem on two antisymmetric matrices by using the coordinate vector $|-y_I^T x^T|$:

$$-\omega^2 \begin{bmatrix} 0 & M \\ -M & -G \end{bmatrix} \begin{bmatrix} -y_I \\ x \end{bmatrix} + \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix} \begin{bmatrix} -y_I \\ x \end{bmatrix} = 0 \tag{25}$$

Equations (23) and (24) can be diagonalized simultaneously when $K>0$ [Eq. (3)]. This diagonalized form, after a rearrangement of the coordinate vector, certainly has the required partitioned structure which at least proves the existence of reducible forms for any gyroscopic system provided $K>0$. This argument, however, does not contain a practical rule for converting a general gyroscopic problem into reducible form as the complete diagonalization of Eq. (23) or (24) solves at the same time as the original problem. A general method starting with a Choleski decomposition of the $2n \times 2n$ system formulation and following an $n \times n$ eigenvalue problem is given by Derksen[7] and Dietrich.[6] An exhaustive discussion of available algorithms to tackle the general gyroscopic problems is given by Dietrich.[6]

Finally, it is easily seen that a previous simultaneous diagonalization of M and K, combined with a rearrangement of the coordinate vector, does not necessarily lead to a reducible system.

Indeed, Eq. (1) becomes

$$E\ddot{q}_I + G_I\dot{q}_I + K_d q_I = 0 \tag{26}$$

with K_d diagonal, $G_I = T^TGT = -G_I^T$, $q = Tq_I$, and T is a nonsingular transformation matrix as given on page 59 of Ref. 3. Equation (26) is reducible if G_I, after rearrangement, can be partitioned

$$G_I = \begin{bmatrix} 0 & g \\ -g^T & 0 \end{bmatrix} \tag{27}$$

with g rectangular $p \times q$ and $p+q=n$.

As G_I stands for a general skew symmetric matrix, it can contain $n(n-1)/2$ different elements, whereas the maximal number $p \cdot q$ for n even is $n^2/4$ and for n odd $(n-1)/4$. From $n \geq 3$ (see Table 1), there is no guarantee to obtain a reducible system, although in Eq. (26) the elements of E and K_d are systematically zero at the places where the element of G_I can be nonzero. Requiring eigenfrequency independent phase relations between the components of q_I in Eq. (27) is sufficient to give Eq. (24) the reducible structure of Eq. (3).

If Eq. (1) is phrased as an eigenvalue problem for the Hermitian impedance matrix:

$$Z = -\omega^2 M + j\omega G + K \qquad Z_q = 0$$

The procedure just described leads to an impedance matrix which contains only real or purely imaginary numbers. It has been shown that such impedance matrices are not necessarily in a reducible form.

References

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